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On the Two-Point Boundary Value Problems using Variational-Fixed Point Iterative Scheme

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ABSTRACT

We developed a new scheme which is obtained by linking finite element,fixed point iterations and variational iteration technique known as variational-fixed point iterative scheme for attacking higher order differential equations. The finite element method is use to obtain the first approximation of VIM, while its the first order approximations is taken as starting value of fixed point iteration method, these sequence give an added advantage over the existing methods. The method is practically applicable to the experiment presented and comparisons were made with exact solution and other methods to demonstrate its reliability, comfortability as well as efficiency of the method.

Keywords: Variational iteration method, Fixed point iteration and Finite element method.

1. Introduction

The approximate solution of bvp of differential equation is of great significance for applied sciences, this led to huge effort being devoted in the area as discussed by Noor and Mohyud-Din (2008a). The vim is an approximation method that has come a very long way since its theory was proposed by He in 1997 as explained in He (1999),Lu (2007). The scheme has been realistic in solving linear, initial as well as boundary value problems as contained Soltani and Shirzadi (2010),He (2000),He and Wu (2007),He (2007). It is valuable to mention that the basis of vim can be referred to Inokuti, (refer Inokuti et al. (2013)), but the true prospective of the scheme was sight-seen by He as argued by Soltani and Shirzadi (2010),He and Wu (2007). It has been a good method for attacking bvp as discussed by He (1999), malleable with good convergence possessions, the method overwhelms some of the drawbacks of other methods by providing good result when rated with analytic solution and other obtainable schemes as in He (1999) ,Ghorbani and Saberi-Nadjafi (2009). Nevertheless, regardless of its proficiency,inadequacies connected with the enactment of the method are noticed, in area of choice of starting value as highlighted by Shang and Han (2010),Noor and Mohyud-Din (2009),Xu (2009).

Conversely, another method of calculating fixed point of repeated function which has been in used for decades is the fixed point iteration method as contained in Stoer and Bulirsch (2013),Berinde (2007), it embroils reiteration of unchanged task beginning with guessing initial value as discussed in Abushammala et al. (2015),Herceg and Krejić (1996),Tatari and Dehghan (2007), which has similar operation to variational iteration method.
Another method of approximation that has been neglected over the years is the finite element method proposed by Clough in 1960 (refer to Cook et al. (2007)), it is a scheme of approximation of boundary value problems used in physics, engineering and technological sciences but has been sidelined because result of closed-form solution that would badge one to scrutinize structure reaction to modify in different constraints is not formed and it only obtain approximate solutions that has intrinsic inaccuracies as discussed by Cook et al. (2007). Among finite element method exists the collocation, galerkin, ritz method and so on.

Research shows that fixed point and variational iteration schemes happens to be good technique of approximation as in Bildik et al. (2013), it was probed that the two technique are alike by adopting the assumption of starting function as studied by Khuri and Sayfy (2014), the construction functional of vim can be found in fixed point iteration as studied by Khuri and Sayfy (2014), Jafari

(2014). This clearify that the two methods interwoven to each other as seen in Jafari (2014).

Driven and stimulated with study in this space, we employed Variational-fixed point iterative scheme for the solution ordinary differential equation of two-point bvp to provide an alternative for determining an initial approximation other than arbitrary choice. This idea has been used by Jafari (2014), Kilicman and Wadai (2016), to solve two-point and three-point boundary value problems endowed by galerkin method. In this work, a collocation method is used as the initial approximation for variational iteration methods while fixed point iterations uses the first order approximation of variational iteration method as its starting function. The collocation methods is numerical for the solution of boundary value problems that uses collocating points within the intervals as explained by Russell and Shampine (1972), Mohsen and El-Gamel (2008) and Viswanadham and Raju (2012).

2. Analysis of the Method

The scheme comprises of collocation technique, fixed point as well as variational iteration techniques, it takes full advantage of these methods but avoids their set backs.

The simple technique of the scheme will be demonstrated by taking equation of the form

$$Pu + Qu = g(x) \tag{1}$$

P and Q are operators while g(t) is forcing term.

The technique can best be described by setting up correct functional as discussed in Driver (2012), He (1997), He (2006) for (1), thus, it follows

$$r_{n+1}(t) = r_n(t) + \int_0^t \lambda(Pr_n(z) + Qr_n(z) - g(z))dz \tag{2}$$

λ is the Langrange multiplier as studied in He (2000), He (2007), Ghorbani and Saberi-Nadjafi (2009) which can be obtained via variational theory , thus

$$\delta r_{n+1}(t) = \delta r_n(t) + \delta \int_0^t \lambda(Pr_n(z) + Qr_n(z) - g(z))dz \tag{3}$$

The Lagranges multiplier can easily be determined by a simple formular as discussed by Ghorbani and Saberi-Nadjafi (2009) which is given as

$$\lambda(t) = \frac{(-1)^h (z-t)^{h-1}}{(h-1)!}$$

Hence, (1) is termed fixed point of the functional from (2) with aid of arbitrary choice of first approximation $u_0(x)$ at $n = 0$ as stated in Jafari and Alipoor (2011), Noor and Mohyud-Din (2008b), Siddiqi and Iftikhar (2015).

We apply the collocation method to determine the initial approximation instead of arbitrary choice as studied by Khuri and Sayfy (2014). After single iteration, a first order approximate (t_1) is obtained; and reiterating the process repeatedly resulted to loftier and burdensome term to control, also the iteration increases the result is moving away or deviate from the exact solution. Therefore, we employ fixed point theory to overwhelmed it.

The Mann Iteration Process: Let (X, D) be a complete metric space and let $\tau : X \to D$. Let $\rho_o \in X$ and let $\rho_{n+1} = f(\tau, \rho_n)$ denotes an iteration procedure which gives a sequence ${\rho_n}_0^\infty$. τ is iteration process defined for arbitrary ρ_0 by

$$\rho_{n+1} = f(\tau, \rho_n) = (1-\alpha_n)\rho_n + \alpha_n \tau \rho_n, n \geq 0 \tag{4}$$

where ${a_n}_{n=1}^\infty$ is a real sequence satisfying $\alpha_n = 1$, $0 \leq \alpha_n \leq 1$ for $n \geq 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$

We let $\rho'' + p(x)\rho' + q(x)\rho = r(x)$ such that $\rho(a) = A; \rho(b) = B, a \leq t \leq b$; where $p, q, r, f \subset [a, b]$,be the given boundary value problem.
Therefore, the scheme

$$\rho''_{n+1} = (1-\lambda_n)\rho''_n + \lambda_n(1-\alpha)\rho''_n$$

where

$$\rho'_{n+1} = \lambda_n(r(x) - p(x)\rho^i_n - q(x)\rho_n) + (1-\lambda_n)\rho^{ii}_n \tag{5}$$

converges for $0 \leq \lambda_n \leq 1$

Justification for convergence of the scheme (5), we can let,

$$\rho'' = f(t, \rho(t), \rho'(t)), \tag{6}$$

$$\rho(a) = A; \rho(b) = B, a \leq t \leq b \tag{7}$$

Therefore, for any $\rho(t) is solution to (6) and (7) of the integral equation on [a, b]$

$$\rho(t) = \int_a^b (G(t,z)f(z, \rho(z), \rho'(z)))dz + w(t) \tag{8}$$

where G(t,z) is a green function of the associated boundary value problem $\rho(a) = A; \rho(b) = B and w(z)$ is a solution of $\rho'' = 0$ that satisfies boundary conditions (7).

Let $\tau : C^1[a_1, a_2] \to C^1[a_1, a_2]$ be defined by

$$(\tau\rho)(t) = \int_{x_1}^{x_2} (G(t,z)f(z, y(z), \rho'(s))dz + w(t) \tag{9}$$

τ is regarded as an operator in which any y(t) is a solution of (6), where τ is termed as fixed point operator.

Now, to prove the equivalence of (4) and (5), we let

$$\rho_{n+1} = (1 - \alpha_n)\rho_n + \alpha_n \tau \rho_n \tag{10}$$

By taking the derivative of (10) to obtained

$$\rho^i_{n+1} = (1 - \alpha_n)\rho^i_n + \alpha_n \tau \rho^i_n$$

We re-differentiate again to get

$$\rho''_{n+1} = (1 - \alpha_n)\rho''_n + \alpha_n \tau \rho^{ii}_n \tag{11}$$

Again we take the derivative (8) twice, to get the following

$$(\tau\rho_n)'(x) = \int_{x_1}^{x_2} \delta/\delta x(G(t,z)f(z, \rho(z), \rho'(z))dz + w(x)'$$

$$(\tau\rho_n)''(x) = \int_{x_1}^{x_2} \delta^2/\delta x^2(G(t,z)f(z,\rho(z),\rho'(s))ds + w(t)'' \tag{12}$$

When we substitute (11) into (12), the following obtained the following equations

$$\rho''_{n+1} = (1-\alpha_n)\rho''_n + \alpha_n \int_{x_1}^{x_2} \delta^2/\delta x^2(G(t,z)f(z,\rho(z),\rho'(z))dz + w(t)'' \tag{13}$$

It is obvious that (11) and (13) becomes

$$\rho''_{n+1} = (1-\alpha_n)\rho''_n + \alpha_n\tau\rho''_n \tag{14}$$

which shows equivalence and convergence of the schemes.

This technique is a tool in approximating boundary value problems iteratively using arbitrary starting function y_0 when $n = 0$. Researchers have found that the variational iteration as well as fixed point iteration methods are similar as argued by Khuri and Sayfy (2014) and Jafari (2014). Then, we can use the first order approximation of vim to determined the initial approximate of fixed point iteration to avoid guessing of starting function. The process can be repeated to achieved convergence of the result.

Collocation method provide an alternative methods for solving boundary value problems without having to find the functional. They are known as residual method R(x) defined as

$$R(x) = y'' + Qy - F$$

where y(x) is approximated by

$$u(x) = v_0(x) + \sum_{i=1}^{n} c_i v_i(x)$$

which is chosen such that $v_0(x)$ satisfies non-homogeneous boundary conditions and $v_i(x)$, $i = 1\cdots n$ satisfy homogeneous boundary conditions and that $v_i(x)$, $i = 0\cdots n$ are linearly independent. On getting the appropriate trial functions we then replace u(x) in the residual R(x) in an attempt to make residuals zero by appropriate choice of coefficient in u(x), which is done at selected points within the interval. The number of points where we set R(x)=0 must be equal to the number of unknown coefficient in u(x) which leads to a set of linear equation that can be solved simultaneously to obtain the parameters for u(x). This gives our approximating solution for the boundary value problems as explained by Ahlberg and Ito (1975).

3. Numerical Analysis

Three examples are considered to demonstrate our methods
Experiment 1. Consider

$$U'' = U' - 4t; 0 \leq t \leq 1 \tag{15}$$

with boundary conditions

$$U(0) = 0, U(1) = 3$$

where

$$y_{exact}(t) = 1.745930121\{1 - e^t\} + 4t + 2t^2$$

Both methods are applied.

1. We first apply Variational Iteration Method: According to VIM, we set up functional

$$r_{n+1}(t) = r_n(t) + \int_0^t \lambda(r_n''(z) - r_n'(z) + 4z)dz \tag{16}$$

and $\lambda(z) = (z - t)$
$n = 0$ in (16) we have

$$r_1(t) = r_0(t) + \int_0^r (z - t)(r_0''(z) - r_0'(z) + 4z)dz \tag{17}$$

The performance of this scheme is on proper selection of first approximation done through guessing. Lets

$$r_0(t) = At + B, \tag{18}$$

be the first approximate, where A and B are constants. We differentiate (18) successively to have 1st and 2nd derivative, plug in (17) to get

$$r_1(t) = At + B + \int_0^t (z - t)(0 - A + 4z)dz \tag{19}$$

and then simplify (19) to obtain

$$r_1(t) = -2/3t^3 + 1/2At^2 + Ax + B \tag{20}$$

Using the conditions in (20), to get A=22/9,B=0, plug in (20), to realize

$$r_1(t) = -2/3t^3 + 11/9t^2 + 22/9t$$

Repeat the procedures to get

$$r_2(t) = -1/6t^4 - 17/60t^3 - 43/20t^2 + 23/10t$$

$$r_3(t) = -1/30t^5 - 89/1230t^4 - 176/615t^3 + 232/205t^2 + 464/205t$$

$$r_4(t) = -1/180t^6 + 1933/37080t^5 - 539/7416t^4 - 539/1854t^3 + 697/618t^2 + 697/309t$$

$$r_5(t) = -1/1260t^7 - 2.424580719 * 10^{-3}t^6 - 0.01454748431t^5 - 3779/51954t^4 - 7558/25977t^3 + 1.127150941t^2 + 19520/8659t$$

$$t_6(x) = -1/10080x^8 - 3.464088676 * 10^{-4}x^7 - 2.424862073 * 10^{-3}x^6 - 0.01454917244x^5 - 0.0727458622x^4 - 0.2909834488x^3 + 1.127049654x^2 + 39041/17320x$$

$$r_7(t) = -1/90720t^9 - 4.330175628 * 10^{-5}t^8 - 3.464140503 * 10^{-4}t^7 - 2.424898352 * 10^{-3}t^6 - 0.01454939011t^5 - 0.07274695056t^4 - 0.2909878022t^3 + 1.127036593t^2 + 1405480/623529t \tag{21}$$

2. Application of the proposed scheme: From VIM, correct functional is set up as

$$r_{n+1}(t) = r_n(t) + \int_0^t \lambda(r_n^{(2)}(z) - r_n^{(1)}(z) + 4z)dz \tag{22}$$

From (22),when $n = 0$

$$r_1(t) = r_0(t) + \int_0^t \lambda(r_0^{(2)}(z) - r_0^{(1)}(z) + 4z)dz \tag{23}$$

we use collocation method to obtain $r_0(x)$. Let u(x) be the collocation approximation for (15). Then

$$u(x) = v_0(x) + \sum_{i=1}^{n} c_i v_i(x)$$

is the basis function in which $v_0(x), v_i$ satisfy non homogeneous and homogeneous condition respectively.

$$u(t) = 3t + C_1(t - t^2) + C_2(t - t^3) \tag{24}$$

Differentiating (24) to get 1st and 2nd derivative, substitute in (15) to get residual

$$-2C_1 - 6C_2t + C_1(1 - 2t) - C_2(1 - 3t^2) = 3 - 4t \tag{25}$$

taking t=1/3 , t=2/3 respectively as collocating point, plug in (25) to give equations

$$7/3C_1 + 8/3C_2 = -5/3 \tag{26}$$

$$5/3C_1 + 11/3C_2 = -1/3 \tag{27}$$

solving (25) and (26) $C_1 = -47/37, C_2 = 18/37$ Plug constants in (24) we have

$$u(t) = 82/37t + 47/37t^2 - 18/37t^3 \tag{28}$$

the approximate solution of (15) used as first approximation. This implies that $r_0(t)$=u(t). Differentiating (28) twice to obtain 1st and 2nd derivative, plug in (23)

$$r_1(t) = 82/37t + 47/37t^2 - 18/37t^3 + \int_0^t (z - t)(12/37 - 202/37z + 54/37z^2 + 4z)dz$$

and simplifying to get

$$r_1(t) = -27/37t^4 + 9/37t^3 + 41/37t^2 + 82/37t \tag{29}$$

When this process is repeated the terms to be integrated is getting loftier and burdensome, also at each iteration step the result deviate from the exact solution, to overcome this challenge we employ the application of fixed point iterative process.

The process is as follows: From (15) We put

$$y''_{n+1}(x) = y'_n(x) - 4x \tag{30}$$

From (30) when $n = 0$, we get

$$y''_1(t) = y'_0(t) - 4t \tag{31}$$

where y_0 is first approximate for fixed point iterative process arbitrary. First order approximates of vim is used as initial guess for FPI. This implies that $y_0 = r_1(t)$, to get

$$y_0(t) = r_1(x) = -27/3xt^4 + 9/37x^3 + 41/37x^2 + 82/37x \tag{32}$$

We differentiate (32) to obtain 1st order derivative then substitute in (31) and simplify to have

$$y''_1(x) = -108/37x^3 + 27/37x^2 - 66/37x + 82/37 \tag{33}$$

To obtain $y_1(x)$, we integrate (33) twice

$$y_1(x) = -27/185x^5 + 9/148x^4 - 11/37x^3 + 41/37x^2 - C_1x + C_2 \tag{34}$$

and imposing the boundary conditions to get the constants of integration $C_1 = 1683/740, C_2 = 0$, Substituting in (34) to get

$$y_1(x) = -27/185x^5 + 9/148x^4 - 11/37x^3 + 41/37x^2 + 1683/740x \tag{35}$$

Repeat the procedures to get

$$y_2(x) = -9/370x^6 + 9/740x^5 - 11/148x^4 - 11/37x^3 + 1683/1480x^2 + 665/296x$$

$$y_3(x) = -9/2590x^7 + 3/1480x^6 - 11/740x^5 - 11/148x^4 - 1277/4440x^3 + 665/592x^2 + 140167/62160x$$

$$y_4(x) = -9/20720x^8 + 3/10360x^7 - 11/4440x^6 - 11/740x^5 - 1277/17760x^4 - 173/592x^3 + 140167/124320x^2 + 23353/10360x$$

$$y_5(x) = -1/20720x^9 + 3/82880x^8 - 11/31080x^7 - 11/4440x^6 - 1277/88800x^5 - 173/2368x^4 - 108473/372960x^3 + 23353/20720x^2 + 525419/233100x$$

$$y_6(x) = -1/207200x^{10} + 1/248640x^9 - 11/248640x^8 - 11/31080x^7 - 11/532800x^6 - 173/11840x^5 - 1084773/1491840x^4 - 6029/20720x^3 + 525419/466200x^2 + 160129/71040x$$

$$y_7(x) = -1/2279200x^{11} + 1/2486400x^{10} - 1/2237760x^9 - 11/248640x^8 - 1277/3729200x^7 - 173/71040x^6 - 108473/7459200x^5 - 6029/82880x^4 - 406981/1398600x^3 + 160129/142080x^2 + 123299453/54700800x \quad (36)$$

Experiment 2. Consider the problem

$$y'' = 2y' - 2y + x + 1; 0 \leq x \leq 1; y(0) = 0, y(1) = 2 \quad (37)$$

Exact equation is given by

$$y_E(x) = e^x(0.8606856797\sin(x) - \cos) + 1/2x + 1$$

We follow the previous procedures.

1. Applying VIM:

We set up a correct functional

$$t_{n+1}(x) = t_n(x) + \int_0^x \lambda(t_n^{(2)}(s) - 2t_n^{(1)}(s) + 2t_n(s) - s - 1)ds \tag{38}$$

when $n = 0$, from in (38) we got

$$t_1(x) = t_0(x) + \int_0^x (s-x)(t_0^{(2)}(s) - 2t_0^{(1)}(s) + 2t_0(s) - s - 1)ds \tag{39}$$

where $t_0(x)$ is the initial approximation which is assumed

$$y_0(x) = ax + b, \tag{40}$$

where a and b constants. Differentiate (40) to get 2nd derivative, plug from (39) at same time evaluate

$$t_1(x) = ax + b + \int_0^x (s-x)(0 - 2a + 2(ax+b) - s - 1)ds \tag{41}$$

Using the conditions in (41), a=$\frac{4}{5}$,b=0 plug in (41), we obtain the first order iterations

$$t_1(x) = -1/10x^3 + 13/10x^2 + 4/5x$$

Repeat the procedures, iterations are obtained

$$t_2(x) = -1/6x^4 + 2/3x^3 + x^2 + 1/2x$$

$$\begin{aligned} t_3(x) = {} & 19/124650x^7 + 493/62325x^6 \\ & - 65/554x^5 + 1/6x^4 + 879/1385x^3 \\ & + 1252/1385x^2 + 1119/2770x \end{aligned}$$

$$\begin{aligned} t_4(x) = {} & -5.596503695 \times 10^{-6}x^9 - 1.95351638 \times 10^{-4}x^8 \\ & + 7.307594061 \times 10^{-7}x^7 - 0.04850342406x^6 \\ & + 4.230956793 \times 10^{-3}x^5 + 1/6x^4 + 0.6243570987x^3 \\ & + 0.8730712962x^2 + 0.3730712962x \end{aligned}$$

$$t_5(x) = -9.711796926 \times 10^{-7}x^{11} + 2.816669887 \times 10^{-6}x^{10} - 2.38701913 \times 10^{-4}x^9 + 3.500838559 \times 10^{-3}x^8 - 0.01392847406x^7 - 9.5986897 \times 10^{-3}x^6 + 4.537264232 \times 10^{-3}x^5 + 0.9911056584x^4 + 0.6212940243x^3 + 0.863882073x^2 + 0.363882073x \quad (42)$$

2. Applying our proposed scheme:

We set up correct functional

$$t_{n+1}(x) = t_n(x) + \int_0^x \lambda(t_n^{(2)}(s) - 2t_n^{(1)}(s) + 2t_n(s) - s - 1)ds \quad (43)$$

From (43), $t_0(x)$ is collocation technique equivalent to u(x) the approximation for (37).Hence,

$$u(x) = v_0(x) + c_1v_1 + c_2v_2$$

where $v_0(x) = 2x, v_1 = (x - x^2), v_2 = (x - x^3)$

$$u(x) = 2x + c_1(x - x^2) + c_2(x - x^3) \quad (44)$$

Differentiating (44) to obtain 1st, 2nd derivative, plug in (37) and simplify, hence residual

$$(-2x^2 + 6x - 4)C_1 + (-2x^3 + 6x^2 + 2x - 2)C_2 = -3x + 5 \quad (45)$$

Taking x=1/3 , x=2/3, plug in (45)

$$-20/9C_1 - 74/27C_2 = 4 \quad (46)$$

$$-8/9C_1 - 70/27C_2 = 3 \quad (47)$$

solving the (46) and (47) gives $C_1 = -\frac{261}{404}$ and $C_2 = -\frac{189}{202}$, then substitute in (44), to get

$$u(x) = (169/404)x + (141/1916)x^2 + (189/202)x^3$$

We lets

$$t_0 = u(x) = (169/404)x + (141/1916)x^2 + (189/202)x^3 \quad (48)$$

Differentiating (48) to get 1st derivative, plug in (43)

$$\begin{aligned} t_1(x) &= (169/404)x + (141/1916)x^2 + (189/202)x^3 \\ &+ \int_0^x (s-x)(261/202 + 267/101x - 2(169/404 + 261/202x \\ &+ 567/202x^2) + 2(169/404x + 261/404x^2 - 189/202x^3) \\ &- x - 1)ds \end{aligned} \tag{49}$$

Simplifying (53) to obtain

$$\begin{aligned} t_1(x) &= (-9/4040)x^7 + (97/8080)x^6 - (193/8080)x^5 \\ &+ (55/2424)x^4 + (189/202)x^3 + (261/404)x^2 \\ &+ (169/404)x \end{aligned} \tag{50}$$

Repeating the process the terms becomes loftier and burdensome, also the values deviates from exact solution as the iteration increase, this motivates the need fixed point iterations procedures.

We put

$$y''_{n+1}(x) = 2y'_n(x) - 2y_n(x) + x + 1 \tag{51}$$

which is from (37), we have y_0 at n=0 the initial approximation for fixed point iteration is taken arbitrary. In this research, first order approximation of VIM is used instead of arbitrary choice. We put $y_0(x) = t_1(x)$, to get

$$\begin{aligned} y_0(x) &= (-9/4040)x^7 + (97/8080)x^6 - (193/8080)x^5 \\ &+ (55/2424)x^4 + (189/202)x^3 + (261/404)x^2 \\ &+ (169/404)x \end{aligned} \tag{52}$$

We differentiate (52) to obtain $1st$ derivative to have

$$\begin{aligned} y'_0(x) &= (-63/4040)x^6 + (582/8080)x^5 - (965/8080)x^4 \\ &+ (220/2424)x^3 + (567/202)x^2 + (522/404)x \\ &+ (169/404) \end{aligned} \tag{53}$$

We substitute (52), (53) in (51) and simplify to obtain

$$\begin{aligned} y''_1(x) &= (9/2020)x^7 - (223/4040)x^6 + (737/808)x^5 \\ &- (689/2424)x^4 - (512/303)x^3 + (873/202)x^2 \\ &+ (555/202)x + 371/202 \end{aligned} \tag{54}$$

We integrate (54) twice to get

$$\begin{aligned} y_1(x) = &(1/16160)x^9 - (223/226240)x^8 + (155/33936)x^7 \\ &- (689/72720)x^6 - (128/1515)x^5 + (291/808)x^4 \\ &+ (185/404)x^3 + (371/404)x^2 + C_1 x + C_2 \end{aligned} \tag{55}$$

and imposing the boundary conditions in (55) to obtain the constant of integrations $C_1 = \frac{144133}{407232}$,$C_2 = 0$, substituting these constant in (55) to have

$$\begin{aligned} y_1(x) = &(1/16160)x^9 - (223/226240)x^8 + (155/33936)x^7 \\ &- (689/72720)x^6 - (128/1515)x^5 + (291/808)x^4 \\ &+ (185/404)x^3 + (371/404)x^2 + 144133/407232x \end{aligned}$$

Repeating the procedures

$$\begin{aligned} y_2(x) = &-(1/888800)x^{11} + (349/10180800)x^{10} - (2113/6108480)x^9 \\ &+ (1507/1018080)x^8 + (67/50904)x^7 - (1897/36360)x^6 \\ &+ (397/4040)x^5 + (23/303)x^4 + (807419/1221696)x^3 \\ &+ (347749/407232)x^2 + (210376/583275)x \end{aligned}$$

$$\begin{aligned} y_3(x) = &(1/69326400)x^{13} - (19/26877312)x^{12} + (601/47995200)x^{11} \\ &- (9353/91627200)x^{10} + (893/3054240)x^9 + (31/14140)x^8 \\ &- (997/50904)x^7 + (1007/36360)x^6 - (87295/2443392)x^5 \\ &+ (229835/1221696)x^4 + (103432079/167983200)x^3 \\ &+ (1004027/1166550)x^2 + (2362977893/6551344800)x \end{aligned}$$

$$\begin{aligned} y_4 = &-(1/7279272000)x^{15} + (601/61145884800)x^{14} \\ &- (7057/26205379200)x^{13} + (10987/3023697600)x^{12} \\ &- (376/15748425)x^{11} + (149/15271200)x^{10} \\ &+ (9449/9162720)x^9 - (107/18180)x^8 + (2466587/256556160)x^7 \\ &- (59743/2443392)x^6 + (364717/26664000)x^5 \\ &+ (55238783/335966400)x^4 + (12189925771/19654034400)x^3 \\ &+ (5638650293/6551344800)x^2 + (16541253503/45859413600)x \end{aligned} \tag{56}$$

$$
\begin{aligned}
y_5 = &+(1/989980992000)x^{17} - (727/7337506176000)x^{16} \\
&+ (10663/2751564816000)x^{15} - (8629/110062592640)x^{14} \\
&+ (2429/2807719200)x^{13} - (12479/3023697600)x^{12} \\
&- (1711/100789920)x^{11} + (5147/15271200)x^{10} \\
&- (14546459/9236021760)x^{9} + (840683/256556160)x^{8} \\
&- (44902841/5879412000)x^{7} - (8065403/1259874000)x^{6} \\
&+ (73594951/196540344000)x^{5} + (297785249/1786730400)x^{4} \\
&+ (85329557399/137578240800)x^{3} + (39470960303/45859413600)x^{2} \\
&+ (57846442359253/160376920704000)x
\end{aligned} \tag{57}
$$

Experiment 3.

$$y'' - y = cosx, 0 \leq x \leq 1 \tag{58}$$

with conditions $y(0) = 0, y(1) = 0.54$

Exact equation is

$$y_e xact(x) = 0.2335726701e^{-x} + 0.2664273299e^{x} - 0.5cosx,$$

According to the VIM a correctional functional can be set up

$$t_{n+1}(s) = t_n(s) + \int_0^s \lambda(t''_n(x) - t_n(x) - cosx)dx \tag{59}$$

$\lambda(s) = (s - x)$

From (59) when $n = 0$, get

$$t_1(x) = t_0(x) + \int_0^x (s - x)(t''_0(s) - t_0(s) - cosx)ds \tag{60}$$

The performance of VIM is on fit choice of initial value

$$t_0(x) = px + q \tag{61}$$

P and q are constants to be determined. We differentiate (61) twice to obtain the second derivative, then substitute in (60)

$$t_1(x) = px + q + \int_0^x (s - x)(0 - ps - q - cos(s))ds \tag{62}$$

we simplify to get

$$t_1(x) = 1/6px^3 + 1/2qx^2 + px - cosx + q + 1 \tag{63}$$

We impose the boundary condition in (63) and simplify to obtain the constants as p=0.06883054791, q=0, then substitutes these constants in (62) to obtain

$$t_1(x) = 0.01147175798x^3 - cosx + 0.06883054791x + 1$$

Repeat the process to

$$t_2(x) = 0.0002836879432x^5 + 0.005673758865x^3 + 0.5x^2 + 0.03404255319x$$

$$t_3(x) = 0.000006522984839x^7 + 0.0002739653632x^5 + 0.04166666667x^4 + 0.005479307265x^3 + 0.03287584359x - cosx + 1 \tag{64}$$

We subject the proposed scheme:

We set up correct functional

$$t_{n+1}(s) = t_n(s) + \int_0^s \lambda(t''_n(x) - t_n(x) - cosx)dx \tag{65}$$

From (65),at n=0 we have $t_0(x)$ which can be used by collocation method. Let u(x) be the approximation for (58).

$$u(x) = v_0(x) + c_1v_1 + c_2v_2$$

where $v_0(x) = 0.54x, v_1 = (x - x^2), v_2 = (x - x^3)$ to yields

$$u(x) = 0.54x + c_1(x - x^2) + c_2(x - x^3) \tag{66}$$

Differentiating (66) successively to get 2nd derivative, plug in (58) and simplify to obtain residual

$$-2C_1 - 6C_2x - (x - x^2)C_1 - (x - x^3)C_2 = 0.54x + cosx \tag{67}$$

Taking x=1/3 , x=2/3 respectively, plug in (77) to get

$$-2.22222C_1 - 2.296296296C_2 = 1.124956946 \tag{68}$$

$$-2.22222C_1 - 4.370370370C_2 = 1.145887261 \tag{69}$$

solving the (68) and (69) gives $C_1 = -0.4958028438$ and $C_2 = -0.01009140188$, then substitute in (66), we get

$$u(x) = 0.03410575432x + 0.4958028438x^2 + 0.01009140188x^3$$

We lets

$$t_0 = U(x) = 0.03410575432x + 0.4958028438x^2 + 0.01009140188x^3 \tag{70}$$

Differentiating (70) to get 2nd derivative, plug in (65)

$$\begin{aligned} t_1(x) &= 0.03410575432x + 0.4958028438x^2 + 0.01009140188x^3 \\ &+ \int_0^x (s-x)(0.9916056876 + 0.06054841128x - (0.03410575432x \\ &+ 0.4958028438x^2 + 0.01009140188x^3) - cosx)ds \end{aligned} \tag{71}$$

Simplifying (75) to obtain

$$\begin{aligned} t_1(x) &= 0.0005045700940x^5 + 0.04131690365x^4 + 0.005684292387x^3 \\ &+ 0.03410575432x - cosx + 1 \end{aligned} \tag{72}$$

Similarly we apply the fixed point iteration. Re-arrange (58), to get

$$y''_{n+1}(x) = y_n(x) + cosx \tag{73}$$

y_0 at n=0 the initial guess for fixed point iteration. In this research first iteration order of VIM is used instead of free choice. Putting $y_0(x) = t_1(x)$,

$$\begin{aligned} y_0(x) &= 0.0005045700940x^5 + 0.04131690365x^4 + 0.005684292387x^3 \\ &+ 0.03410575432x - cosx + 1 \end{aligned} \tag{74}$$

We substitute (78) in (77) and simplify to obtain

$$\begin{aligned} y''_1(x) &= 0.0005045700940x^5 + 0.04131690365x^4 + 0.005684292387x^3 \\ &+ 0.03410575432x - cosx + 1 + cosx \end{aligned} \tag{75}$$

We integrate (79) twice to get

$$\begin{aligned} y_1(x) &= 0.0000120135736x^7 + 0.001377230122x^6 \\ &+ 0.0002842146194x^5 + 0.005684292387x^3 \\ &+ 0.5x^2 + C_1x + C_2 \end{aligned} \tag{76}$$

and imposing the boundary conditions in (80) to obtain the constant of integrations $C_1 = 0.0326422493$,$C_2 = 0$, substituting these constant in (80) to have

$$\begin{aligned} y_1(x) &= 0.0000120135736x^7 + 0.001377230122x^6 \\ &+ 0.0002842146194x^5 + 0.005684292387x^3 \\ &+ 0.5x^2 + 0.0326422493x \end{aligned}$$

Repeating the procedures,

$$\begin{aligned} y_2(x) &= 1.668551899 * 10^{-7}x^9 + 0.00002459339504x^8 \\ &+ 0.000006767014747x^7 + 0.0416666668x^4 \\ &+ 0.005440374883x^3 + 0.0328795224x - cosx + 1 \end{aligned}$$

$$\begin{aligned} y_3(x) &= 1.516865363 * 10^{-9}x^{11} + 2.732599449 * 10^{-7}x^{10} \\ &+ 9.398631593 * 10^{-8}x^9 + 0.000000676014747x^7 \\ &+ 0.00138888889x^6 + 0.0002720187442x^5 \\ &+ 0.0054799204x^3 + 0.5x^2 + 0.0328520362x \end{aligned}$$

Remarks: $y_e(x)$, $y_v(x)$ and $y_n(x)$ are exact, variational iteration and the new method at 7th,5th and 3rd iteration as shown on; Table 1 to Table 3 respectively:

Table 1: Experiment 1

x	$y_e(x)$	$y_v(x)$	$y_n(x)$	$\|y_e(x)-y_v(x)\|$	$\|y_e(x)-y_n(x)\|$
0.1	0.236378926	0.2363792742	0.2363789646	3.482×10^{-7}	3.86×10^{-8}
0.2	0.493446256	0.4934469879	0.4934463010	7.319×10^{-7}	4.5×10^{-8}
0.3	0.769170969	0.7691721269	0.7691709883	1.1579×10^{-6}	1.93×10^{-8}
0.4	1.0161308446	1.016130071	1.0161349744	1.625×10^{-6}	4.1298×10^{-5}
0.5	1.367377993	1.367380129	1.367377909	2.136×10^{-6}	8.4×10^{-8}
0.6	1.684638024	1.684640681	1.684637901	2.657×10^{-6}	1.23×10^{-7}
0.7	2.010056614	2.01005661729	2.010058481	3.115×10^{-6}	1.33×10^{-7}
0.8	2.340291179	2.340294453	2.340291073	3.274×10^{-6}	1.06×10^{-7}
0.9	2.671634964	2.671637569	2.671634909	2.605×10^{-6}	5.5×10^{-8}
1.0	3.000000000	2.999999999	2.999999999	1.0×10^{-9}	1.0×10^{-9}

Table 2: Experiment 2

x	$y_e(x)$	$y_v(x)$	$y_n(x)$	$\|y_e(x)-y_v(x)\|$	$\|y_e(x)-y_n(x)\|$
0.1	0.0453123570	0.04566502314	0.045312927	3.527×10^{-4}	5.704×10^{-7}
0.2	0.1117938936	0.1125693845	0.1117950742	7.755×10^{-4}	1.181×10^{-6}
0.3	0.2037672269	0.2050401541	0.2037687318	1.273×10^{-3}	1.505×10^{-6}
0.4	0.3259485570	0.3277900023	0.3259498430	1.842×10^{-3}	1.270×10^{-7}
0.5	0.4834305629	0.4858861799	0.4834311235	2.456×10^{-3}	5.606×10^{-7}
0.6	0.6816531912	0.6846778350	0.6816528289	3.025×10^{-3}	3.623×10^{-7}
0.7	0.9263602997	0.9298186829	0.9263592125	3.458×10^{-3}	1.087×10^{-6}
0.8	1.2235399880	1.2269645270	1.2235386600	3.425×10^{-3}	1.328×10^{-6}
0.9	1.5793463100	1.5818433510	1.5793453430	2.497×10^{-3}	9.670×10^{-7}
1.0	2.0000000000	2.0000000001	2.0000000000	1.00×10^{-10}	0.0

Table 3: Experiment 3

x	$y_e(x)$	$y_v(x)$	$y_n(x)$	$\|y_e(x)-y_v(x)\|$	$\|y_e(x)-y_n(x)\|$
0.1	0.0082909459	0.008293067368	0.00829068765	2.121468×10^{-6}	2.5825×10^{-7}
0.2	0.0266149147	0.026619179950	0.02661442262	4.26525×10^{-6}	4.9208×10^{-7}
0.3	0.0550059233	0.055001237233	0.05500524370	6.44903×10^{-6}	6.796×10^{-7}
0.4	0.0935008170	0.093509501360	0.09350001489	8.68436×10^{-6}	8.0211×10^{-7}
0.5	0.1421421107	0.142153051800	0.14214212635	1.09411×10^{-5}	8.472×10^{-7}
0.6	0.2009818384	0.200994907700	0.20098102870	1.30693×10^{-5}	8.097×10^{-7}
0.7	0.2700864185	0.270101054600	0.27008572640	1.46361×10^{-5}	6.921×10^{-7}
0.8	0.3495425383	0.349557178400	0.34954220334	1.46401×10^{-5}	5.049×10^{-7}
0.9	0.4394640663	0.439475099600	0.43946380030	1.10333×10^{-5}	2.66×10^{-7}
1.0	0.5399999997	0.539999999800	0.54000000000	1.00×10^{-10}	3.00×10^{-10}

4. Discussion

In this work, we applied the variational-fixed point iterative scheme on boundary value problems of ordinary differential equation. The illustrative experiments that we examined, shows that the variational-fixed point iterative scheme is reliable, accurate and efficient as seen with other method, it is clearly shown in Table 1,2 and 3. It is shown clearly that after few iterations, the method have upper hand over the existing ones. It is also clear that the convergence of the result increases as the number of the iteration increases.

5. Conclusion

We developed a new scheme which is an alternative schemes for solving Two-Point Boundary Value problems which is effective and reliable as revealed by the results of the experiments presented from the tables. We conclude that the variational-fixed point iterative technique is a very powerful and efficient techniques for finding the analytic solutions of boundary value problems of ordinary differential equations.

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